**EXERCISES [MAI 5.18-5.19]**

**DIFFERENTIAL EQUATIONS**

**SOLUTIONS**

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## A. Paper 1 questions (SHORT)

### D.E. OF SEPARABLE VARIABLES

**1.** (a) $\frac{dy}{dx} = 2x + 5 \Rightarrow y = x^2 + 5x + c$

$y(0) = 3 \Rightarrow c = 3$

Therefore, $y = x^2 + 5x + 3$

(b) $\frac{dy}{dx} = \sin x + \cos x \Rightarrow y = -\cos x + \sin x + c$

$y(0) = 3 \Rightarrow -1 + c = 3 \Rightarrow c = 4$

Therefore, $y = \sin x - \cos x + 4$

**2.** $y = e^x - x^2 + C$

$3 = e^0 - 0 + C \Rightarrow 3 = 1 + C \Rightarrow C = 2$

Therefore, $y = e^x - x^2 + 2$

**3.** $\frac{dy}{dx} = \frac{2x}{3y^2} \Rightarrow 3y^2 dy = 2x dx \Rightarrow \int 3y^2 dy = \int 2x dx$

$\Rightarrow y^3 = x^2 + c$

$y(0) = 2 \Rightarrow c = 8$

Therefore, $y^3 = x^2 + 8 \Rightarrow y = \sqrt[3]{x^2 + 8}$

**4.** (a) $\frac{dy}{dx} = 10x^4 \Rightarrow y = 2x^5 + c$

$y(1) = 2 \Rightarrow 2 + c = 2 \Rightarrow c = 0$

Therefore, $y = 2x^5$

(b) $\frac{dy}{dx} = 10x^4 y^2 \Rightarrow \frac{dy}{y^2} = 10x^4 dx \Rightarrow \int \frac{dy}{y^2} = \int 10x^4 dx \Rightarrow -\frac{1}{y} = 2x^5 + c$

$y(1) = 2 \Rightarrow 2 + c = -\frac{1}{2} \Rightarrow c = -2.5$

Therefore, $-\frac{1}{y} = 2x^5 - 2.5 \Rightarrow y = -\frac{1}{2x^5 - 2.5}$

(c) $\frac{dy}{dx} = 10x^4 y \Rightarrow \frac{dy}{y} = 10x^4 dx \Rightarrow \int \frac{dy}{y} = \int 10x^4 dx \Rightarrow \ln y = 2x^5 + c$

$y(1) = 2 \Rightarrow 2 + c = \ln 2 \Rightarrow c = \ln 2 - 2$

Therefore, $\ln y = 2x^5 + \ln 2 - 2 \Rightarrow y = e^{2x^5 + \ln 2 - 2} \Rightarrow y = 2e^{2x^5 - 2}$

**5.** $\int \frac{dy}{y^2} = \int 2x dx \Rightarrow \ -\frac{1}{y} = x^2 + c$

Using $y(0) = 1$ gives $C = -1$ $\left(-\frac{1}{y} = x^2 - 1\right)$

$y = -\frac{1}{x^2 - 1}$ $\left(= \frac{1}{1 - x^2}\right)$

**6.** $xy\ \frac{dy}{dx} = 1 + y^2 \Rightarrow \int \frac{y}{1+y^2} dy = \int \frac{1}{x} dx$

$\frac{1}{2}\ln(1 + y^2) = \ln x + \ln c \Rightarrow \ln(1 + y^2) = 2\ln x + 2\ln c \Rightarrow \ln(1 + y^2) = \ln x^2 c^2$

$1 + y^2 = kx^2$ $(k = c^2)$

$y = 0$ when $x = 2$, and so $1 = 4k$

Thus, $y^2 = \frac{1}{4}x^2 - 1$ (or $x^2 - 4y^2 = 4$)

**7.**

$\frac{dy}{y} = \frac{x\,dx}{x^2 + 1}$

$\ln y = \frac{1}{2}\ln(x^2 + 1) + A$

$\ln 1 = \frac{1}{2}\ln 2 + A$ for substituting $x = 1$, $y = 1$

$A = -\frac{1}{2}\ln 2$

$\ln y = \frac{1}{2}\ln(x^2 + 1) - \frac{1}{2}\ln 2$

$\ln y = \ln\sqrt{\frac{x^2 + 1}{2}}$

$y = \sqrt{\frac{x^2 + 1}{2}}$

**8.** (a) $\frac{1}{y} + \frac{1}{1-y} = \frac{1 - y + y}{y(1-y)} = \frac{1}{y - y^2}$

(b) $\frac{dy}{dt} = k\,(y - y^2) \Rightarrow \frac{dy}{y - y^2} = k dt \Rightarrow \int \frac{dx}{y - y^2} = \int k dt$

$\Rightarrow \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int k dt$

$\Rightarrow \ln y - \ln(1 - y) = kt + c \Rightarrow \ln\frac{y}{1-y} = kt + c \Rightarrow \frac{y}{1-y} = e^{kt+c}$

Solve for $y$: $y = e^{kt+c} - ye^{kt+c} \Rightarrow y(1 + e^{kt+c}) = e^{kt+c} \Rightarrow y = \frac{e^{kt+c}}{1 + e^{kt+c}}$

Divide both terms by $e^{kt+c}$: $y = \frac{1}{\frac{1}{e^{kt+c}} + 1} = \frac{1}{1 + e^{-kt-c}} = \frac{1}{1 + Ae^{-kt}}$

**9.** If $kx = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$ Then $\int kx\,\mathrm{d}x = \int mv\,\mathrm{d}v$

$\Rightarrow \dfrac{1}{2}kx^2 = \dfrac{1}{2}mv^2 + C$

When $x = 0$, $v = v_0$, therefore $C = -\dfrac{1}{2}mv_0^2$ Therefore $v^2 = v_0^2 + \dfrac{kx^2}{m}$

Therefore when $x = 2$, $v = \sqrt{v_0^2 + \dfrac{4k}{m}}$

**10.** From the diagram,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1} \Rightarrow \int \frac{\mathrm{d}y}{y} = \int \mathrm{d}x \Rightarrow \ln y = x + c \Rightarrow y = \mathrm{e}^{x+c} = A\mathrm{e}^x.$$

But R (0,2) lies on the curve and so $A = 2$.

Thus $y = 2\mathrm{e}^x$

**11.** (a) Given $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -kv \Leftrightarrow \int \dfrac{\mathrm{d}v}{v} = -k\int \mathrm{d}t$

$\Leftrightarrow \ln v = -kt + C$

$\Leftrightarrow v = A\mathrm{e}^{-kt} (A = \mathrm{e}^C)$

At $t = 0$, $v = v_0 \Rightarrow A = v_0 \Leftrightarrow v = v_0\mathrm{e}^{-kt}$

(b) Put $v = \dfrac{v_0}{2}$ then $\dfrac{v_0}{2} = v_0\mathrm{e}^{-kt}$

$\Leftrightarrow \dfrac{1}{2} = \mathrm{e}^{-kt} \Leftrightarrow \ln\dfrac{1}{2} = -kt$

$\Leftrightarrow t = \dfrac{\ln 2}{k}$

(c) $v_0 = 3.6$

$\dfrac{\ln 2}{k} = 5 \Rightarrow k = \dfrac{\ln 2}{5}$

$v = 3.6\mathrm{e}^{-\frac{\ln 2}{5}t}$ or $v = 3.6\mathrm{e}^{-0.139t}$

(d) For $t = 24$, $v = 0.129$ units$^3$

**12.** (a) If $M$ g is the material present at time t, then $\dfrac{\mathrm{d}M}{\mathrm{d}t} = kM$ where $k$ is a constant.Then,

$\dfrac{\mathrm{d}M}{M} = k\mathrm{d}t \Rightarrow \ln M = kt + c$

$\Rightarrow M = \mathrm{e}^{kt+c} = A\mathrm{e}^{kt}$

(b) When $t = 0$, $A = 50$

When $t = 10$, $M = 48 \Rightarrow 48 = 50\mathrm{e}^{10k}$.

$k = \dfrac{\ln 0.96}{10} = -0.00408$

For half life, $25 = 50\mathrm{e}^{kt} \Rightarrow \ln 0.5 = kt \Rightarrow t = \dfrac{10\ln 0.5}{\ln 0.96} = 169.8$.

Therefore, half-life = 170 years (3 s.f.)

## EULER'S METHOD

**13.** (a) $x_{n+1} = x_n + 0.1$, $y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$

| n | x | y |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.1 | 1.1 |
| 2 | 0.2 | 1.222 |
| 3 | 0.3 | 1.3753284 |
| 4 | 0.4 | 1.573481221 |

approximate value of $y$ = 1.57

(b) the approximate value is less than the actual value because it is assumed that $\frac{dy}{dx}$ remains constant throughout each interval whereas it is actually an increasing function

**14.** $x_{n+1} = x_n + 0.1$, $y_{n+1} = y_n + 0.1(e^{x_n} + 2y_n^2)$

| n | x | y |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.1 | 1.3 |
| 2 | 0.2 | 1.7485170918 |
| 3 | 0.3 | 2.482119772 |
| 4 | 0.4 | 3.849289365 |

required approximation = 3.85

**15.** (a) $x_{n+1} = x_n + 0.25$, $y_{n+1} = y_n + 0.25(\frac{y_n^2 + x_n^2}{2x_n^2})$

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | –1 |
| 1 | 1.25 | –0.75 |
| 2 | 1.5 | –0.58 |
| 3 | 1.75 | –0.436311 |

$\Rightarrow y(2) \approx -0.304$

**16.** (a) (i) $x_{n+1} = x_n + 0.1$, $y_{n+1} = y_n + 0.1(2x_n(1 + x_n^2 - y_n))$

| n | $x_i$ | $y_i$ |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 1.1 | 2 |
| 2 | 1.2 | 2.0462 |
| 3 | 1.3 | 2.1407 |

$f(1.3) = 2.14$

(ii) Decrease the step size

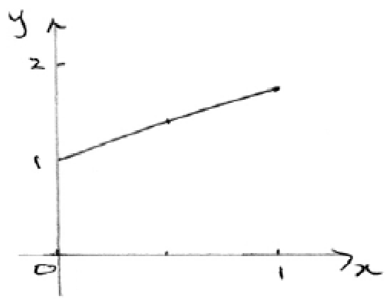
**17.** (a) $x_{n+1} = x_n + 0.25$, $y_{n+1} = y_n + 0.25\left(1 - \dfrac{x_n y_n}{4 - x_n^{\,2}}\right)$

| $n$ | $x$ | $y$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.25 | 1.25 |
| 2 | 0.5 | 1.48015873 |
| 3 | 0.75 | 1.680820106 |
| 4 | 1 | 1.839139009 |

To two decimal places, when $x = 1$, $y = 1.84$.

(b) $y(1) = 1.77$, so $1.84 > 1.77$

(c)



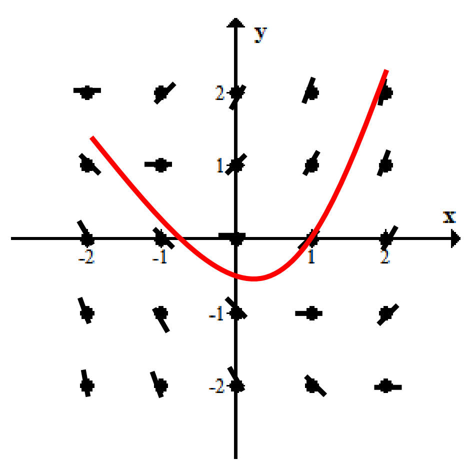

Since the function is concave down the value of $y$ is over-estimated at each step.

**SLOPE FIELDS**

**18.** (a) $\dfrac{dy}{dx} = x + y$

| | | | | | | |
|---|---|---|---|---|---|---|
| $y$ | **2** | 0 | 1 | 2 | 3 | 4 |
| | **1** | -1 | 0 | 1 | 2 | 3 |
| | **0** | -2 | -1 | 0 | 1 | 2 |
| | **-1** | -3 | -2 | -1 | 0 | 1 |
| | **-2** | -4 | -3 | -2 | -1 | 0 |
| $\dfrac{dy}{dx}$ | | **-2** | **-1** | **0** | **1** | **2** |
| | | $x$ | | | | |

(b) and (c)



## B. Paper 2 questions (*LONG*)

**19.** (a) $x^2+y^2=1 \Rightarrow y=\pm\sqrt{1-x^2}$

For $y=\sqrt{1-x^2}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2x}{2\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}=-\frac{x}{y}$$

For $y=-\sqrt{1-x^2}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{-2x}{2\sqrt{1-x^2}}=\frac{x}{\sqrt{1-x^2}}=-\frac{x}{-\sqrt{1-x^2}}=-\frac{x}{y}$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y} \Rightarrow y\mathrm{d}y=-x\mathrm{d}x \Rightarrow \int y\mathrm{d}y=-\int x\mathrm{d}x$

$\Rightarrow \frac{y^2}{2}=-\frac{x^2}{2}+c \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}=c$

$\Rightarrow x^2+y^2=C$

For $x=0, y=1 \Rightarrow c=1$.

Hence $x^2+y^2=1$

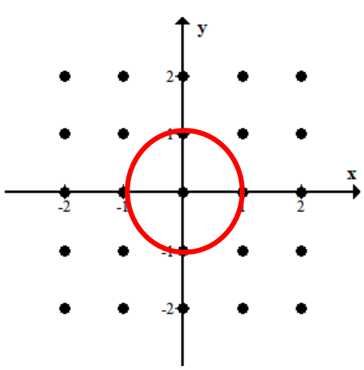
(c) $x_{n+1}=x_n+0.01$,

$y_{n+1}=y_n-0.01\frac{x_n}{y_n}$ (50 steps)

$y(0.5)=0.86918\ldots \cong 0.869$

Percentage error = $\left|\frac{0.86918-\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}\right|\times 100\% = 0.36\%$

(d) The particular solution is the unit circle:



**20.** (a) the solution (by any method is $y = \frac{C}{x}$ where $C$ is a constant.

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + c \Rightarrow \ln yx = c \Rightarrow yx = e^c$$

$$y = \frac{e^c}{x} \Rightarrow y = \frac{C}{x}$$

For $x = 1, y = 2 \Rightarrow C = 2$. Hence $y = \frac{2}{x}$

(b) For $x = 2$, $y = 1$

(c) (i) $x_{n+1} = x_n + 0.2$, $y_{n+1} = y_n - 0.2\frac{y_n}{x_n}$

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 1.2 | 1.6 |
| 2 | 1.4 | 1.33333 |
| 3 | 1.6 | 1.14285 |
| 4 | 1.8 | 1 |
| 5 | 2 | 0.88888 |

For $x = 2$, $y \cong 0.88888... \cong 0.889$

(ii) $x_{n+1} = x_n + 0.01$, $y_{n+1} = y_n - 0.01\frac{y_n}{x_n}$ (100 steps) $y \cong 0.99497... \cong 0.995$